Variations of Gauss-Codazzi-Ricci Equations in Kaluza-Klein Reduction (String Theory) and Cauchy Problem (General Relativity)

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Abstract

We find a kind of variations of Gauss-Codazzi-Ricci equations suitable for Kaluza-Klein reduction and Cauchy problem. Especially the counterpart of extrinsic curvature tensor has antisymmetric part as well as symmetric one. If the dependence of metric tensor on reduced dimensions is negligible it becomes a pure antisymmetric tensor. PACS:03.70;11.15
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1 Introduction

1.1 Introductory Remarks

As is well-known Gauss-Codazzi-Ricci equations are very important instruments for describing a submanifold in a Riemann space. By nature they appear in the Cauchy problem of general relativity [1], [2].However the celebrated Kaluza-Klein dimensional reduction in string/M theory involves also a submanifold in a higher dimensional space-time. Therefore it is interesting to find the variations of Gauss-Codazzi-Ricci equations suitable for dimensional reduction. It would be best to look for a formulation which is convenient for both purposes: Cauchy problem in general relativity and Kaluza-Klein reduction in string/M theory.
There are authors who proposed to use the lapse function[3] and shift vector function[4] for characterizing the Cauchy problem as well as Hamiltonian formulation in general relativity. As shown in refs.[1, 2] n-dimensional (pseudo) Riemann space can be foliated by a family of n-1 dimensional Cauchy surfaces. On these hypersurfaces there exists "normal vector field" which is connected with the extrinsic curvature. But because of the shift vector the covariant form $n_A (A = 1, \cdots n)$ of normal vector is not in the same direction with contravariant vector n^A. They like the pair of axes of a oblique coordinates, n_A is orthogonal to the initial surface($x^n = 0$) but n^A is not (see next subsection for the detail).

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Now we suggest a similar construction with lapse-like function related to the dilaton fields in string theory and shift-like vector function connected with Kaluza-Klein gauge field. Of course in general relativity we can use them to determine the Hamiltonian structure as well. In this construction the role of n_A and n^A are exchanged. The vector orthogonal to the initial surface is n^A instead of n_A.
Significantly, the new construction leads to variations of Gauss-Codazzi-Ricci equations. Especially the counterpart of the extrinsic curvature has antisymmetric part and symmetric one. If the dependence of metric tensor on reduced dimensions can be neglected it becomes a pure antisymmetric tensor. This is what we will depict in this paper.
In second part of this section we would like to give a brief review about lapse function and shift vector in order to compare them with new construction. We restrict ourselves from sec.2 to 4 only one reduced dimension included. Section 2 is devoted for Gauss and Weingarten formulae while Gauss and Codazzi equations are presented in sec.3 and sec.4 respectively. In sec.5 we discuss this construction presented in general relativity. Sec.6 collects the main interest in the dimensional reduction including reduced dimensions more than one. Finally a short discussion about future work will appear in sec.7. Besides we list the relevant Christoffel symbols in an Appendix.

1.2 Lapse Function and Shift Vector Review[2],[5]

Einstein's theory asserts that space-time structure and gravitation are described by a pseudo Riemann space and a metric tensor g_{AB}. It has been proved that n-dimensional (pseudo) Riemann space can be foliated by a family of n-1 dimensional Cauchy surfaces. On these hypersurfaces there exists a normal vector field n_A satisfying the normalized condition

$$n_A n^A = \epsilon = \pm 1, \tag{1}$$

+1 and -1 corresponding to the space-like and time-like vector respectively. When the lapse function N and shift function $N_\alpha (\alpha = 1 \cdots m)$ (in sec.2-4 m=n-1) are introduced, using a coordinate frame, vector n_A can be denoted by its components

$$n_A = (\underbrace{0, \cdots, 0}_{n-1}, \epsilon N), \tag{2}$$

$$n^A = (-N^\alpha/N, 1/N). \tag{3}$$

Let h_{AB} represent the metric on each hypersurfaces induced by g_{AB}. Then we have

$$g_{AB} = h_{AB} + \epsilon\, n_A n_B, \tag{4}$$

$$h_{AB} n^B = h^{AB} n_B = 0. \tag{5}$$

Explicitly we can write down inverse metric

$$g^{AB} = \begin{pmatrix} h^{\alpha\beta} + \epsilon\, N^\alpha N^\beta / N^2 & -\epsilon\, N^\alpha/N^2 \\ -\epsilon\, N^\beta/N^2 & \epsilon/N^2 \end{pmatrix}, \tag{6}$$

or

$$h^{AB} = \begin{pmatrix} h^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \tag{7}$$

and others. It is worthy to note that we have $h^n_A = 0$ which is the most remarkable distinction with later construction.
The extrinsic curvature tensor is defined by

$$K_{AB} = h^\alpha_A h^\beta_B \, K_{\alpha\beta}, \tag{8}$$

$$K_{\alpha\beta} = \epsilon \, N \Gamma^n_{\alpha\beta} = \frac{\epsilon}{2N^2}[(\nabla_\alpha \, N_\beta + \nabla_\beta \, N_\alpha) - \partial_n h_{\alpha\beta}], \tag{9}$$

$$\nabla_\alpha \, N_\beta \equiv \partial_\alpha - \mathbf{\Gamma}^\gamma_{\alpha\beta} \, N_\gamma \tag{10}$$

which is a symmetric tensor connected with the second fundamental form of the hypersurface, Γ^C_{AB} and $\mathbf{\Gamma}^\gamma_{\alpha\beta}$ are n-dimensional and m(=n-1)-dimensional Christoffel symbol respectively.
From about definitions one can derive the Gauss-Codazzi equations and solve the Einstein equation in vacuum through Hamiltonian formulation either. Please find the detail in refs[2],[5].

2 Gauss Formula and Weingarten Formula

The standard Kaluza Klein reduction from n-dimensional space-time to n-1 dimensional subspace is shown in the following formula

$$ds^2_n = e^{2a\phi} ds^2_{n-1} + e^{2b\phi}(dx^n + \mathcal{A}_\alpha \, dx^\alpha)^2, \tag{11}$$

$$ds^2_{n-1} = h_{\alpha\beta} dx^\alpha \, dx^\beta \tag{12}$$

in which ϕ is the dilaton field and $\mathcal{A}_\alpha$is a gauge field. Apart from a conformal factor $e^{2a\phi}$ the metric is in the form

$$ds^2 = g_{AB} dx^A \, dx^B = h_{\alpha\beta} dx^\alpha \, dx^\beta + \frac{\epsilon}{N^2}(dx^n + N_\alpha \, dx^\alpha)^2, \tag{13}$$

where

$$N_\alpha = \mathcal{A}_\alpha \qquad N^{-2} = e^{2(b-a)\phi} \equiv \, e^{2\mathbf{b}\phi}. \tag{14}$$

In eq.(13) we have added a factor ϵ so that it can fit Cauchy problem as well. We also introduce the normal vector n^A such that

$$g_{AB} = h_{AB} + \epsilon \, n_A \, n_B \quad n_A n^A = \epsilon \quad h_{AB} n^B = h^{AB} n_B = 0, \tag{15}$$

and denote it in its components form

$$n^A = (0, \cdots, \epsilon \, N), \qquad n_A = (N_\alpha/N, 1/N). \tag{16}$$

Similar to the lapse-shift case we can write down inverse metric

$$g^{AB} = \begin{pmatrix} h^{\alpha\beta} & -N^{\alpha} \\ -N^{\beta} & N_{\gamma}\, N^{\gamma} + \epsilon\, N^2 \end{pmatrix} = h^{AB} + \begin{pmatrix} 0 & 0 \\ 0 & \epsilon\, N^2 \end{pmatrix} \tag{17}$$

and

$$h_{AB} = \begin{pmatrix} h_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}. \tag{18}$$

It is easy to find that

$$h_{n\alpha} = h_{\alpha\, n} = h^{\alpha}_{n} = 0, \qquad h^{\beta}_{\alpha} = \delta^{\beta}_{\alpha}. \tag{19}$$

We may imitate ref.[2] to introduce the "time flow" vector field

$$t_A = (0, \cdots 0, 1), \qquad t^A = (-N^{\alpha}, N_{\gamma}\, N^{\gamma} + \epsilon\, N^2),$$

so that

$$n_A t^A = n^A t_A = \epsilon\, N,$$

and

$$h_{AB} t^B = -N_{\alpha}.$$

But we have no chance to use it latter.
From eqs.(15)-(18) we know that

$$h_{\alpha\beta} = h^{A}_{\alpha}\, h^{B}_{\beta}\, g_{AB}. \tag{20}$$

Differentiate it we get

$$\partial_{\gamma}\, h_{\alpha\beta} = (\partial_{\gamma}\, h^{A}_{\alpha}) h_{A\beta} + (\partial_{\gamma}\, h^{B}_{\beta}) h_{\alpha\, B} + h^{A}_{\alpha} h^{B}_{\beta} h^{C}_{\gamma} \partial_C g_{AB} - h^{A}_{\alpha} h^{B}_{\beta} h^{n}_{\gamma} \partial_n g_{AB}. \tag{21}$$

Because of eq.(19) it becomes

$$D_{\gamma}\, h_{\alpha\beta} \equiv \partial_{\gamma}\, h_{\alpha\beta} + h^{n}_{\gamma} \partial_n h_{\alpha\beta} = h^{A}_{\alpha} h^{B}_{\beta} h^{C}_{\gamma} \partial_C g_{AB}. \tag{22}$$

Define that

$$P^{\gamma}_{\alpha\beta} = \frac{1}{2} h^{\gamma\delta} (D_{\alpha}\, h_{\delta\beta} + D_{\beta}\, h_{\alpha\delta} - D_{\delta}\, h_{\alpha\beta}) \equiv \mathbf{\Gamma}^{\gamma}_{\alpha\beta} + H^{\gamma}_{\alpha\beta}, \tag{23}$$

i.e.

$$H^{\gamma}_{\alpha\beta} \equiv -\frac{1}{2} h^{\gamma\delta} (N_{\alpha} \partial_n h_{\delta\beta} + N_{\beta} \partial_n h_{\alpha\delta} - N_{\delta} \partial_n h_{\alpha\beta}). \tag{24}$$

Insert eq.(22) into eq.(23) we obtain

$$\begin{aligned} & P^{\gamma}_{\alpha\beta} = \\ & \quad \frac{1}{2} h^{\gamma\delta} h^{D}_{\delta} h^{A}_{\alpha} h^{B}_{\beta} (\partial_A g_{DB} + \partial_B g_{AD} - \partial_D g_{AB}) = \\ & \quad h^{\gamma}_{C}\, h^{A}_{\alpha} h^{B}_{\beta} \frac{1}{2} g^{CD} (\partial_A g_{DB} + \partial_B g_{AD} - \partial_D g_{AB}) = \\ & \quad h^{\gamma}_{C}\, h^{A}_{\alpha} h^{B}_{\beta} \Gamma^{C}_{AB} = \end{aligned}$$

$$h^\gamma_C(h^A_\alpha h^B_\beta \Gamma^C_{AB} + \partial_\alpha\, h^C_\beta). \tag{25}$$

In last step we have used the ambiguity because of eq.(19). It is clear that

$$h^\gamma_C \tilde{\nabla}_\alpha\, h^C_\beta \equiv\; h^\gamma_C(\partial_\alpha\, h^C_\beta + \Gamma^C_{AB} h^A_\alpha h^B_\beta - P^\delta_{\alpha\beta}\, h^C_\delta) = 0, \tag{26}$$

which tells us that $\tilde{\nabla}_\alpha\, h^C_\beta$ is proportional to the normal vector field n^C, hence we can define a tensor $K_{\alpha\beta}$ by

$$\tilde{\nabla}_\alpha\, h^C_\beta = K_{\alpha\beta} n^C \tag{27}$$

that is the Gauss formula in present case. Operator $\tilde{\nabla}_\alpha$ we have introduced is an operator which operates on the n-dimensional index as well as m-dimensional index simultaneously[6]. In fact if we define the following operations

$$\tilde{\nabla}_\beta\, n^C = \partial_\beta\, n^C + \Gamma^C_{BA} h^B_\beta n^A, \qquad \tilde{\nabla}_\beta\, n_C = \partial_\beta\, n_C - \Gamma^A_{BC} h^B_\beta n_A; \tag{28}$$

and

$$\tilde{\nabla}_\beta\, u_\gamma = \partial_\beta\, u_\gamma - P^\alpha_{\beta\gamma}\, u_\alpha, \qquad \tilde{\nabla}_\beta\, u^\gamma = \partial_\beta\, u^\gamma + P^\gamma_{\beta\alpha}\, u^\alpha. \tag{29}$$

Operator $\tilde{\nabla}_\alpha$ on h^C_β certainly agrees with eq.(26). However we have to note that

$$\begin{aligned}
&\tilde{\nabla}_\beta\, g_{AB} = \\
&\quad \partial_\beta\, g_{AB} - \Gamma^C_{DA} h^D_\beta\, g_{CB} - \Gamma^C_{DB} h^D_\beta\, g_{CA} = \\
&\quad -\Gamma^C_{nA} h^n_\beta g_{CB} - \Gamma^C_{nB} h^n_\beta g_{CA} \neq 0.
\end{aligned} \tag{30}$$

From definition eq.(23) and eq.(24) we have

$$\begin{aligned}
&\tilde{\nabla}_\gamma\, h_{\alpha\beta} = \\
&\quad \partial_\gamma\, h_{\alpha\beta} - P^\delta_{\gamma\beta}\, h_{\alpha\delta} - P^\delta_{\alpha\gamma}\, h_{\delta\beta} = \\
&\quad \nabla_\gamma\, h_{\alpha\beta} - H^\delta_{\gamma\beta}\, h_{\alpha\delta} - H^\delta_{\alpha\gamma}\, h_{\delta\beta} = \\
&\quad -h^n_\gamma \partial_n h_{\alpha\beta}
\end{aligned} \tag{31}$$

or

$$\begin{aligned}
&\tilde{\nabla}_\gamma\, h_{\alpha\beta} = \\
&\quad h^A_\alpha h^B_\beta h^C_\gamma \partial_C g_{AB} - h^\delta_C\, h\gamma^A h^B_\beta \Gamma^C_{AB} h_{\alpha\delta} - \\
&\quad h^A_\alpha h^C_\gamma \Gamma^D_{AC} h^B_\beta (g_{DB} - \epsilon\, n_D\, n_B) - h^n_\gamma \partial_n h_{\alpha\beta} = \\
&\quad h^A_\alpha h^B_\beta h^C_\gamma (\partial_C g_{AB} - \Gamma^D_{CA} g_{DB} - \Gamma^D_{CB} g_{AD}) - h^n_\gamma \partial_n h_{\alpha\beta} = \\
&\quad h^A_\alpha h^B_\beta h^C_\gamma \nabla_C g_{AB} - h^n_\gamma \partial_n h_{\alpha\beta} = \\
&\quad -h^n_\gamma \partial_n h_{\alpha\beta}.
\end{aligned} \tag{32}$$

Eq.(25)leads to

$$\begin{aligned}
&\tilde{\nabla}_\alpha h^C_\beta = \\
&\quad \partial_\alpha h^C_\beta + \Gamma^C_{AB} h^A_\alpha h^B_\beta - P^\gamma_{\alpha\beta} h^C_\gamma = \\
&\quad (\delta^C_D - h^\gamma_D h^C_\gamma)(\partial_\alpha h^D_\beta + \Gamma^D_{AB} h^A_\alpha h^B_\beta) = \\
&\qquad \epsilon\, n_D(\partial_\alpha h^D_\beta + \Gamma^D_{AB} h^A_\alpha h^B_\beta) n^C
\end{aligned} \tag{33}$$

therefore

$$\begin{aligned}
&K_{\alpha\beta} = \\
&\quad \epsilon\, n_D(\partial_\alpha h^D_\beta + \Gamma^D_{AB} h^A_\alpha h^B_\beta) = \\
&\quad -\frac{\epsilon}{N}\partial_\alpha N_\beta + \frac{\epsilon}{N}(\Gamma^n_{AB} + N_\gamma \Gamma^\gamma_{AB}) h^A_\alpha h^B_\beta = \\
&\quad -\frac{\epsilon}{N}\partial_\alpha N_\beta + \frac{\epsilon}{N}(\Gamma^n_{\alpha\beta} + N_\gamma \Gamma^\gamma_{\alpha\beta}) \\
&\quad -\frac{\epsilon N_\alpha}{N}(\Gamma^n_{\beta n} + N_\gamma \Gamma^\gamma_{\beta n}) - \frac{\epsilon N_\beta}{N}(\Gamma^n_{\alpha n} \\
&\quad + N_\gamma \Gamma^\gamma_{\alpha n}) + \frac{\epsilon}{N} N_\alpha N_\beta (\Gamma^n_{nn} + N_\gamma \Gamma^\gamma_{nn}) = \\
&\qquad -\frac{\epsilon}{2N}\mathcal{F}_{\alpha\beta} - \frac{1}{2N}\mathcal{H}_{\alpha\beta}.
\end{aligned} \tag{34}$$

in which

$$\mathcal{F}_{\alpha\beta} \equiv \partial_\alpha N_\beta - \partial_\beta N_\alpha, \qquad \mathcal{H}_{\alpha\beta} \equiv N^2 \partial_n h_{\alpha\beta} + \epsilon \partial_n (N_\alpha N_\beta). \tag{35}$$

Similar to subsection Lapse review we can also define

$$K_{AB} = h^\alpha_A h^\beta_B K_{\alpha\beta}, \qquad K = K^A_A = K^\alpha_\alpha. \tag{36}$$

Here we want to point out that the tensor K_{AB} has not only a symmetric part ($\mathcal{H}$) but also an antisymmetric part ($\mathcal{F}$). Since

$$0 = \tilde{\nabla}_\beta (n^A n_A) = (\tilde{\nabla}_\beta n^A) n_A + n^A \tilde{\nabla}_\beta n_A \tag{37}$$

$$= n_B n_A \tilde{\nabla}_\beta g^{AB} + 2n^A \tilde{\nabla}_\beta n_A \tag{38}$$

$$= 2\Gamma^A_{nB} h^n_\beta n^B n_A + 2n^A \tilde{\nabla}_\beta n_A \tag{39}$$

so that

$$n^A \tilde{\nabla}_\beta n_A = -\Gamma^A_{nB} h^n_\beta n^B n_A, \tag{40}$$

we then obtain the Weingarten formula as following

$$\begin{aligned}
&\tilde{\nabla}_\beta n_A = \\
&\quad h^B_A \tilde{\nabla}_\beta n_B - \epsilon\, n_A \Gamma^B_{nC} h^n_\beta n^C n_B = \\
&\quad h^\alpha_A h^B_\alpha (\partial_\beta n_B - \Gamma^D_{BC} h^C_\beta n_D) - \epsilon\, n_A \Gamma^B_{nC} h^n_\beta n^C n_B = \\
&\quad -h^\alpha_A (\partial_\beta h^D_\alpha + \Gamma^D_{BC} h^B_\alpha h^C_\beta) n_D - \epsilon\, n_A \Gamma^B_{nC} h_\beta{}^n n^C n_B =
\end{aligned}$$

$$-\epsilon\, h_A^{\alpha}\, K_{\beta\alpha} - \epsilon \Gamma^B_{nC} h^n_\beta n^C n_B n_A, \tag{41}$$

or

$$\begin{aligned}
&\tilde{\nabla}_\beta\, n^A = \\
&\quad -\epsilon\, h^A_\alpha K_\beta{}^\alpha + (\Gamma^A_{nB} n^B + \Gamma^C_{nB} h^{AB} n_C) h^n_\beta = \\
&\quad -\epsilon\, h^A_\alpha K_\beta{}^\alpha + (\frac{1}{N} h^{A\alpha} \partial_n N_\alpha + \mathbf{b} n^A \partial_n \phi) h^n_\beta.
\end{aligned} \tag{42}$$

3 Variation of Gauss Equation

To get the Gauss equation we have to calculate $\tilde{\nabla}_\gamma \tilde{\nabla}_\alpha\, h^C_\beta$, that is

$$\begin{aligned}
&\tilde{\nabla}_\gamma \tilde{\nabla}_\alpha\, h^C_\beta = \\
&\quad \partial_\gamma \partial_\alpha\, h^C_\beta + (\partial_\alpha \Gamma^C_{AB}) h^A_\alpha h^B_\beta h^D_\gamma - \\
&\quad (\partial_n \Gamma^C_{AB}) h^A_\alpha h^B_\beta h^n_\gamma + \Gamma^C_{AB} (\partial_\gamma\, h^A_\alpha) h^B_\beta \\
&\quad + \Gamma^C_{AB} h^A_\alpha \partial_\gamma\, h^B_\beta - (\partial_\gamma\, P^\delta_{\alpha\beta}) h^C_\delta - \\
&\quad P^\delta_{\alpha\beta} \partial_\gamma\, h^C_\delta + \Gamma^C_{DF} h^D_\gamma (\partial_\alpha\, h^F_\beta + \\
&\quad \Gamma^F_{AB} h^A_\alpha h^B_\beta - P^\delta_{\alpha\beta}\, h^F_\delta) - (\partial_\alpha\, h^C_\delta + \\
&\quad \Gamma^C_{AB} h^A_\alpha h^B_\delta - P^\eta_{\alpha\delta}\, h^C_\eta) P^\delta_{\gamma\beta} - (\partial_\delta\, h^C_\beta + \\
&\quad \Gamma^C_{AB} h^A_\delta h^B_\beta - P^\eta_{\delta\beta}\, h^C_\eta) P^\delta_{\gamma\alpha}.
\end{aligned} \tag{43}$$

From the definition of Riemann tensor we find

$$\begin{aligned}
&(\tilde{\nabla}_\gamma \tilde{\nabla}_\alpha - \tilde{\nabla}_\alpha \tilde{\nabla}_\gamma) h^C_\beta = \\
&\quad h^A_\alpha h^B_\beta h^D_\gamma (\partial_D \Gamma^C_{AB} - \partial_A \Gamma^C_{DB} + \Gamma^C_{DF} \Gamma^F_{AB} - \Gamma^C_{AF} \Gamma^F_{DB}) - h^C_\delta (\partial_\gamma\, P^\delta_{\alpha\beta} - \partial_\alpha\, P^\delta_{\gamma\beta} - \\
&\quad P^\delta_{\alpha\eta}\, P^\eta_{\gamma\beta} + P^\delta_{\gamma\eta}\, P^\eta_{\alpha\beta}) - (\partial_n \Gamma^C_{AB}) h^B_\beta (h^A_\alpha h^n_\gamma - h^A_\gamma h^n_\alpha) + \Gamma^C_{AB} h^B_\beta (\partial_\gamma\, h^A_\alpha - \\
&\quad \partial_\alpha\, h^A_\gamma) = h^A_\alpha h^B_\beta h^D_\gamma R_{ADB}{}^C - h^C_\delta S_{\alpha\gamma\beta}{}^\delta - (\partial_n \Gamma^C_{\alpha\, B} h^n_\gamma - \partial_n \Gamma^C_{\gamma\, B} h^n_\alpha) h^B_\beta + \\
&\quad \Gamma^C_{nB} h^B_\beta (\partial_\gamma\, h^n_\alpha - \partial_\alpha\, h^n_\gamma).
\end{aligned} \tag{44}$$

In which we have used a symbol

$$\begin{aligned}
&S_{\alpha\gamma\beta}{}^\delta \equiv \partial_\gamma\, P^\delta_{\alpha\beta} - \partial_\alpha\, P^\delta_{\gamma\beta} - P^\delta_{\alpha\eta}\, P^\eta_{\gamma\beta} + P^\delta_{\gamma\eta}\, P^\eta_{\alpha\beta} = \\
&\quad \mathbf{R}_{\alpha\gamma\beta}{}^\delta + \partial_\gamma\, H^\delta_{\alpha\beta} - \partial_\alpha\, H^\delta_{\gamma\beta} + H^\delta_{\gamma\eta}\, H^\eta_{\alpha\beta} - H^\delta_{\alpha\eta}\, H^\eta_{\gamma\beta} + \\
&\quad \Gamma^\delta_{\gamma\eta}\, H^\eta_{\alpha\beta} + \Gamma^\eta_{\alpha\beta}\, H^\delta_{\gamma\eta} - \Gamma^\delta_{\alpha\eta}\, H^\eta_{\gamma\beta} - \Gamma^\eta_{\gamma\beta}\, H^\delta_{\alpha\eta}.
\end{aligned} \tag{45}$$

Tensor $S_{\alpha\gamma\beta}{}^{\delta}$ has the following symmetric properties like Riemann tensor

$$S_{\alpha\gamma\beta}{}^{\delta} = -S_{\gamma\alpha\beta}{}^{\delta} \tag{46}$$

and

$$S_{\alpha\gamma\beta}{}^{\delta} + S_{\gamma\beta\alpha}{}^{\delta} + S_{\beta\alpha\gamma}{}^{\delta} = 0 \tag{47}$$

but have no corresponding symmetries after contracting with a metric tensor. It is easy to find also

$$\tilde{\nabla}_\alpha h^\gamma_B = K_\alpha{}^\gamma n_B - h_{B\beta} h^n_\alpha \partial_n h^{\gamma\beta} - h^{C\gamma}\Gamma^A_{nC} h^n_\alpha g_{AB} - h^\gamma_A \Gamma^A_{nB} h^n_\alpha. \tag{48}$$

As a result we obtain

$$\begin{aligned}
& h^\zeta_C\, h^A_\alpha h^B_\beta h^D_\gamma R_{ADB}{}^C - S_{\alpha\gamma\beta}{}^\zeta + h^\zeta_C\, h^B_\beta \Gamma^C_{nB}(\partial_\gamma\, h^n_\alpha - \partial_\alpha\, h^n_\gamma) - h^\zeta_C\, h^B_\beta (h^n_\gamma \partial_n \Gamma^C_{\alpha\, B} - h^n_\alpha \partial_n \Gamma^C_{\gamma\, B}) = \\
&\quad h^\zeta_C (\tilde{\nabla}_\gamma \tilde{\nabla}_\alpha - \tilde{\nabla}_\alpha \tilde{\nabla}_\gamma) h^C_\beta = \\
&\quad \tilde{\nabla}_\alpha\, h^\zeta_C \tilde{\nabla}_\gamma\, h^C_\beta - \tilde{\nabla}_\gamma\, h^\zeta_C \tilde{\nabla}_\alpha\, h^C_\beta = \\
&\quad K_{\gamma\beta} n^C (K_\alpha{}^\zeta\, n_C - h_{C\delta} h^n_\alpha \partial_n h^{\zeta\delta} - h^{B\zeta}\Gamma^A_{nB} h^n_\alpha g_{AC} - h^\zeta_A \Gamma^A_{nC} h^n_\alpha) - K_{\alpha\beta} n^C (K_\gamma{}^\zeta\, n_C - \\
&\quad h_{C\delta} h^n_\gamma \partial_n h^{\zeta\delta} - h^{B\zeta}\Gamma^A_{nB} h^n_\gamma g_{AC} - h^\zeta_A \Gamma^A_{nC} h^n_\gamma) = \\
&\qquad \epsilon\, K_{\gamma\beta} K_\alpha{}^\zeta - \epsilon\, K_{\alpha\beta} K_\gamma{}^\zeta - (h^n_\alpha K_{\gamma\beta} - h^n_\gamma K_{\alpha\beta})(h^\zeta_A\, n^B + h^{B\zeta} n_A)\Gamma^A_{nB}.
\end{aligned} \tag{49}$$

Contracting γ with ζ, and multiplying $h^{\alpha\beta}$, then eq.(49) becomes

$$\begin{aligned}
& h^{AB} h^{CD} R_{ADBC} = R - 2\epsilon\, R_{ab} n^a n^b = R - 2\epsilon\, N^2 R_{nn} = \\
&\quad \mathbf{R} + h^{\alpha\beta}\partial_\gamma\, H^\gamma_{\alpha\beta} - h^{\alpha\beta}\partial_\alpha\, H^\gamma_{\gamma\beta} + \mathbf{\Gamma}^\gamma_{\gamma\eta}\, h^{\alpha\beta} H^\eta_{\alpha\beta} - \\
&\quad 2h^{\alpha\beta}\mathbf{\Gamma}^\gamma_{\alpha\eta}\, H^\eta_{\gamma\beta} + h^{\alpha\beta}\mathbf{\Gamma}^\eta_{\alpha\beta}\, H^\gamma_{\gamma\eta} + H^\gamma_{\gamma\eta}\, h^{\alpha\beta} H^\eta_{\alpha\beta} - \\
&\quad h^{\alpha\beta} H^\gamma_{\alpha\eta}\, H^\eta_{\gamma\beta} - h^{B\alpha}\Gamma^\gamma_{nB}\mathcal{F}_{\alpha\gamma} + h^{B\alpha}(N_\alpha \partial_n \Gamma^\gamma_{\gamma\, B} - N_\gamma \partial_n \Gamma^\gamma_{\alpha\, B}) - \\
&\quad \frac{\epsilon}{4N^2}\mathcal{F}_{\alpha\gamma}\mathcal{F}^{\alpha\gamma} + \frac{\epsilon}{4N^2}(\mathcal{H}_{\alpha\gamma}\mathcal{H}^{\alpha\gamma} - \mathcal{H}^2) - \\
&\quad \frac{1}{2N^2}[N_\alpha(\epsilon\mathcal{F}^{\beta\alpha} + \mathcal{H}^{\beta\alpha}) - N^\beta\mathcal{H}]\partial_n N_\beta = \\
&\quad \mathbf{R} - \frac{3\epsilon}{4N^2}\mathcal{F}_{\alpha\gamma}\mathcal{F}^{\alpha\gamma} + \frac{3\epsilon}{N^2}\mathcal{F}^{\alpha\gamma} N_\alpha \partial_n N_\gamma + N_\alpha \partial^\gamma\, h^{\alpha\beta}\partial_n h_{\beta\gamma} + \\
&\quad h^{\alpha\beta}(\partial^\gamma\, N_\gamma \partial_n h_{\alpha\beta} - \partial^\gamma\, N_\alpha \partial_n h_{\beta\gamma}) + N^\alpha\, h^{\gamma\delta}\partial_\alpha \partial_n h_{\gamma\delta} + \\
&\quad \frac{\epsilon}{4N^2}(\mathcal{H}_{\alpha\gamma}\mathcal{H}^{\alpha\gamma} - \mathcal{H}^2) + \frac{1}{N^2}(N^\alpha\, N_\gamma \partial_n \mathcal{H}^\gamma_\alpha - N_\alpha\, N^\alpha \partial_n \mathcal{H}) - \\
&\quad \frac{1}{2N^2}(N_\alpha\, N^\alpha \mathcal{H}^{\gamma\delta} - N^\gamma\, N^\delta \mathcal{H})\partial_n h_{\gamma\delta} + \frac{2\mathbf{b}}{N^2}(N^\alpha\, N^\gamma \mathcal{H}_{\alpha\gamma} - N_\gamma\, N^\gamma \mathcal{H})\partial_n \phi + \\
&\qquad \frac{1}{4} h^{\alpha\beta}(2N^\gamma\, N^\delta - h^{\gamma\delta} N_\lambda\, N^\lambda)(\partial_n h_{\alpha\beta}\partial_n h_{\gamma\delta} - \partial_n h_{\alpha\delta}\partial_n h_{\beta\gamma})
\end{aligned} \tag{50}$$

in which we have used n-1 dimensional harmonic condition $h^{\alpha\beta}\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = 0$. On the other hand, we can calculate the second term on the left hand side of the equation

$$
\begin{aligned}
&2\epsilon\, R_{ab}n^a n^b = \\
&\quad \frac{\epsilon}{2N^2}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} - \frac{2\epsilon}{N^2}\mathcal{F}^{\alpha\beta}N_\alpha\partial_n N_\beta - \\
&\quad \frac{\epsilon}{2N^2}(\mathcal{H}_{\alpha\beta}\mathcal{H}^{\alpha\beta} - \mathcal{H}^2) + \frac{2}{N^2}\mathcal{H}^{\alpha\beta}\, N_\alpha\partial_n N_\beta - \frac{2}{N^2}\mathcal{H}N^\alpha\partial_n N_\alpha + \\
&\qquad\qquad 2\epsilon[\nabla_C(n^A\nabla_A n^C) - \nabla_A(n^A\nabla_C n^C)].
\end{aligned} \tag{51}
$$

Therefore

$$
\begin{aligned}
R \;=\; & \mathbf{R} - \frac{\epsilon}{4N^2}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} + \frac{\epsilon}{N^2}\mathcal{F}^{\alpha\beta}N_\alpha\partial_n N_\beta \\
& + N_\alpha\partial^\gamma\, h^{\alpha\beta}\partial_n h_{\beta\gamma} + h^{\alpha\beta}(\partial^\gamma\, N_\gamma\partial_n h_{\alpha\beta} \\
& - \partial^\gamma\, N_\alpha\partial_n h_{\beta\gamma}) + N^\alpha\, h^{\gamma\delta}\partial_\alpha\partial_n h_{\gamma\delta} \\
& + \frac{3\epsilon}{4N^2}(\mathcal{H}_{\alpha\beta}\mathcal{H}^{\alpha\beta} - \mathcal{H}^2) - \frac{1}{2N^2}(N_\gamma\, N^\gamma + 2\epsilon\, N^2)\mathcal{H}^{\alpha\beta}\partial_n h_{\alpha\beta} + \\
& \frac{\epsilon}{2N^2}(N^\alpha\, N^\gamma + 2N^2 h^{\alpha\gamma})\mathcal{H}\partial_n h_{\alpha\beta} + \frac{1}{N^2}(N^\alpha\, N_\gamma\partial_n\mathcal{H}^\gamma_\alpha \\
& - N_\alpha\, N^\alpha\partial_n\mathcal{H}) + \frac{2\mathbf{b}}{N^2}(N^\alpha\, N^\gamma\mathcal{H}_{\alpha\gamma} - N_\alpha\, N^\alpha\mathcal{H})\partial_n\phi + \\
& \frac{1}{4}h^{\alpha\beta}(2N^\gamma\, N^\delta - h^{\gamma\delta}N_\lambda\, N^\lambda)(\partial_n h_{\alpha\beta}\partial_n h_{\gamma\delta} - \\
& \quad \partial_n h_{\alpha\delta}\partial_n h_{\beta\gamma}) + 2\epsilon[\nabla_C(n^A\nabla_A n^C) - \nabla_A(n^A\nabla_C n^C)].
\end{aligned} \tag{52}
$$

The last divergence term in above equation, which may be neglected in Hamiltonian formulation [7][2], is important in the dimensional reduction. So we now evaluate it as well

$$
\begin{aligned}
&\nabla_C(n^A\nabla_A n^C) - \nabla_A(n^A\nabla_C n^C) = \\
&\quad \epsilon\nabla_C[N(\partial_n n^C + \epsilon\, N\Gamma^C_{nn})] - \epsilon\, N\nabla_C n^C\Gamma^\alpha_{\alpha\, n} - \epsilon\partial_n(N\nabla_C n^C) - \epsilon\, N\nabla_C n^C\Gamma^n_{nn} = \\
&\quad \partial_\alpha(N^2\Gamma^\alpha_{nn}) + \partial_n(N\partial_n N + N^2\Gamma^n_{nn}) + \partial_n[-\mathcal{H}/2 + N^2 N_\alpha\Gamma^\alpha_{nn} + \epsilon\, N^\beta(\Gamma^n_{n\beta} \\
&\quad + N_\alpha\Gamma^\alpha_{n\beta}) - \epsilon\, N_\gamma\, N^\gamma(\Gamma^n_{nn} + N_\alpha\Gamma^\alpha_{nn})] + (N\partial_n N + N^2\Gamma^n_{nn})(\Gamma^\alpha_{\alpha\, n} + \\
&\quad \Gamma^n_{nn}) + N^2(\Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{nn} + \Gamma^n_{n\beta}\Gamma^\beta_{nn}) + [-\mathcal{H}/2 + N^2 N_\alpha\Gamma^\alpha_{nn} + \epsilon\, N_\beta(\Gamma^n_{n\beta} + \\
&\qquad\qquad N_\alpha\Gamma^\alpha_{n\beta}) - \epsilon\, N_\gamma\, N^\gamma(\Gamma^n_{nn} + N_\alpha\Gamma^\alpha_{nn})](\Gamma^\alpha_{\alpha\, n} + \Gamma^n_{nn}).
\end{aligned} \tag{53}
$$

From Appendix we easily find

$$
\Gamma^\alpha_{\alpha\beta} + \Gamma^n_{n\beta} = \mathbf{\Gamma}^\alpha_{\alpha\beta} + \mathbf{b}\partial_\beta\phi \qquad \Gamma^\alpha_{\alpha\, n} + \Gamma^n_{nn} = \mathbf{b}\partial_\beta\phi + \frac{1}{2}h^{\alpha\beta}\partial_n h_{\alpha\beta} \tag{54}
$$

$$\Gamma^n_{n\beta} + N_\alpha \Gamma^\alpha_{n\beta} = \mathbf{b}\partial_\beta\phi \qquad \Gamma^n_{nn} + N_\alpha \Gamma^\alpha_{nn} = -\frac{1}{N}\partial_n N = \mathbf{b}\partial_n\phi. \tag{55}$$

Eventually we obtain

$$\begin{aligned}
&\nabla_C(n^A\nabla_A n^C) - \nabla_A(n^A\nabla_C n^C) = \\
&\quad -\epsilon\mathbf{b}(\partial^\alpha\partial_\alpha\phi + \mathbf{b}\partial^\alpha\phi\partial_\alpha\phi) + \epsilon\partial^\alpha\partial_n N_\alpha + \epsilon\mathbf{b}(2\partial^\alpha\phi\partial_n N_\alpha + \partial^\alpha N_\alpha\partial_n\phi \\
&\quad +2N^\alpha\partial_\alpha\partial_n\phi + 2\mathbf{b}N^\alpha\partial_\alpha\phi\partial_n\phi - N^\alpha\partial^\beta\phi\partial_n h_{\alpha\beta} + \frac{1}{2}N^\gamma\partial_\gamma\phi\, h^{\alpha\beta}\partial_n h_{\alpha\beta}) \\
&\quad -\partial_n(\mathcal{H}/2 + \epsilon\mathbf{b}N_\gamma\, N^\gamma\partial_n\phi) - (\mathcal{H}/2+ \\
&\qquad\qquad \epsilon\mathbf{b}N_\gamma\, N^\gamma\partial_n\phi)(\mathbf{b}\partial_n\phi + \frac{1}{2}h^{\alpha\beta}\partial_n h_{\alpha\beta})
\end{aligned} \tag{56}$$

in which n-1 dimensional harmonic condition is also used.

4 Variation of Codazzi Equation

First of all, let us calculate the double covariant derivative of normal vector n^C

$$\begin{aligned}
&\tilde\nabla_\gamma\tilde\nabla_\beta\, n^C = \\
&\quad \partial_\gamma\partial_\beta\, n^C + (\partial_D\Gamma^C_{BA})h^B_\beta h^D_\gamma n^A - (\partial_n\Gamma^C_{BA})h^B_\beta h^n_\gamma n^A + \Gamma^C_{BA}(\partial_\gamma\, h^B_\beta)n^A \\
&\quad +\Gamma^C_{BA}h^B_\beta\partial_\gamma\, n^A + \Gamma^C_{DA}h^D_\gamma\partial_\beta\, n^A + \Gamma^C_{DA}\Gamma^A_{BF}h^D_\gamma h^B_\beta n^F - P^\alpha_{\gamma\beta}\partial_\alpha\, n^C - \\
&\qquad\qquad P^\alpha_{\gamma\beta}\Gamma^C_{AF}h^A_\alpha n^F.
\end{aligned} \tag{57}$$

Next, we have to antisymmetrize it as follows

$$\begin{aligned}
&(\tilde\nabla_\gamma\tilde\nabla_\beta - \tilde\nabla_\beta\tilde\nabla_\gamma)n^C = \\
&\quad (\partial_D\Gamma^C_{BA} - \partial_B\Gamma^C_{DA})h^B_\beta h\gamma^D n^A + (\Gamma^C_{DF}\Gamma^F_{BA} - \Gamma^C_{BF}\Gamma^F_{DA})h^D_\gamma h^B_\beta n^A \\
&\quad +\Gamma^C_{BA}(\partial_\gamma\, h^B_\beta - \partial_\beta\, h^B_\gamma)n^A - (\partial_n\Gamma^C_{BA})(h^B_\beta h^n_\gamma - h^B_\gamma h^n_\beta)n^A = \\
&\quad R_{BDA}{}^C h^B_\beta h\gamma^D n^A + \epsilon\, N(\partial_\gamma\, h^n_\beta - \partial_\beta\, h^n_\gamma)\Gamma^C_{nn} - (\partial_n\Gamma^C_{BA})(h^B_\beta h^n_\gamma \\
&\qquad\qquad -h^B_\gamma h^n_\beta)n^A.
\end{aligned} \tag{58}$$

By using of the Weingarten formula (42) the left hand side of eq.(58) becomes

$$\begin{aligned}
&(\tilde\nabla_\gamma\tilde\nabla_\beta - \tilde\nabla_\beta\tilde\nabla_\gamma)n^C = \\
&\quad -\epsilon\, h^C_\alpha(\tilde\nabla_\gamma\, K_\beta{}^\alpha - \tilde\nabla_\beta\, K_\gamma{}^\alpha) - \epsilon(K_\beta{}^\alpha\, K_{\gamma\alpha} - K_\gamma{}^\alpha\, K_{\beta\alpha})n^C + (\frac{1}{N}h^{C\alpha}\partial_n N_\alpha \\
&\quad +\mathbf{b}n^C\partial_n\phi)(\partial_\gamma\, h^n_\beta - \partial_\beta\, h^n_\gamma) + \partial_\gamma(\frac{1}{N}h^{C\alpha}\partial_n N_\alpha + \mathbf{b}n^C\partial_n\phi)h^n_\beta - \partial_\beta(\frac{1}{N}h^{C\alpha}\partial_n N_\alpha
\end{aligned}$$

$$+\mathbf{b}n^C\partial_n\phi)h^n_\gamma + (\Gamma^C_{AD}h^A_\gamma h^n_\beta - \Gamma^C_{AD}h^A_\beta h^n_\gamma)(\frac{1}{N}h^{D\alpha}\partial_n N_\alpha + \mathbf{b}n^D\partial_n\phi) \quad (59)$$

multiplying h^α_C on both sides of eq.(58), we would see

$$\begin{aligned}
&-\epsilon(\tilde{\nabla}_\gamma K_\beta{}^\alpha - \tilde{\nabla}_\beta K_\gamma{}^\alpha) = \\
&\quad R_{BDA}{}^C h^\alpha_C h^B_\beta h^D_\gamma n^A + \epsilon N h^\alpha_C(\partial_\gamma h^n_\beta - \partial_\beta h^n_\gamma)\Gamma^C_{nn} + \epsilon N h^\alpha_C(h^n_\beta\partial_n\Gamma^C_{\gamma n} \\
&\quad -h^n_\gamma\partial_n\Gamma^C_{\beta n}) - \frac{1}{N}h^{\alpha\delta}\partial_n N_\delta(\partial_\gamma h^n_\beta - \partial_\beta h^n_\gamma) - \partial_\gamma(\frac{1}{N}h^{\alpha\delta}\partial_n N_\delta)h^n_\beta \\
&\quad +\partial_\beta(\frac{1}{N}h^{\alpha\delta}\partial_n N_\delta)h^n_\gamma - h^\alpha_C\Gamma^C_{AD}(h^A_\gamma h^n_\beta - \\
&\qquad\qquad h^A_\beta h^n_\gamma)(\frac{1}{N}h^{D\delta}\partial_n N_\delta + \mathbf{b}n^D\partial_n\phi). \qquad (60)
\end{aligned}$$

Contracting index γ with α we achieve the goal.

$$\begin{aligned}
&-\epsilon(\tilde{\nabla}_\alpha K^\alpha_\beta - \tilde{\nabla}_\beta K) = \\
&\quad R_{BA}h^B_\beta n^A + \epsilon N(\partial_\alpha h^n_\beta - \partial_\beta h^n_\gamma)\Gamma^\alpha_{nn} + \epsilon N(h^n_\beta\partial_n\Gamma^\alpha_{\alpha n} - h^n_\alpha\partial_n\Gamma^\alpha_{\beta n}) \\
&\quad -\frac{1}{N}h^{\gamma\alpha}\partial_n N_\alpha(\partial_\gamma h^n_\beta - \partial_\beta h^n_\gamma) - \partial_\alpha(\frac{1}{N}h^{\alpha\gamma}\partial_n N_\gamma)h^n_\beta + \\
&\quad \partial_\beta(\frac{1}{N}h^{\alpha\gamma}\partial_n N_\gamma)h^n_\alpha - (\Gamma^\alpha_{\alpha\gamma}h^n_\beta - \Gamma^\alpha_{\beta\gamma}h^n_\alpha)(\frac{1}{N}h^{\gamma\delta}\partial_n N_\delta) \\
&\quad +(\Gamma^\alpha_{\alpha n}h^n_\beta - \Gamma^\alpha_{\beta n}h^n_\alpha)(\frac{1}{N}N^\gamma\partial_n N_\gamma - \epsilon\mathbf{b}N\partial_n\phi) = \\
&\quad R_{BA}h^B_\beta n^A + \frac{\mathbf{b}}{N}\mathcal{F}_{\alpha\beta}\partial^\alpha\phi - \frac{3\mathbf{b}}{2N}N^\alpha\mathcal{F}_{\alpha\beta}\partial_n\phi \\
&\quad +\frac{1}{2N}N_\alpha\mathcal{F}_{\beta\gamma}\partial_n h^{\alpha\gamma} - \frac{1}{2N}N^\gamma(\partial_\beta\partial_n N_\gamma + \partial_\gamma\partial_n N_\beta) + \\
&\quad \frac{1}{N}h^{\alpha\gamma}N_\beta\partial_\alpha\partial_n N_\gamma - \frac{\mathbf{b}}{N}(N^\alpha\partial_\alpha\phi\partial_n N_\beta \\
&\quad +\partial_\beta\phi N^\alpha\partial_n N_\alpha - 2N_\beta\partial^\alpha\phi\partial_n N_\alpha) - \frac{1}{N}N_\alpha(\partial_\beta h^{\alpha\delta} + h^{\gamma\delta}\boldsymbol{\Gamma}^\alpha_{\beta\gamma})\partial_n N_\delta \\
&\quad +\frac{\epsilon}{2N}(N^\alpha\partial_n\mathcal{H}_{\alpha\beta} - N_\beta h^{\alpha\gamma}\partial_n\mathcal{H}_{\alpha\gamma}) \\
&\quad +\frac{\epsilon}{4N^3}(N^\gamma\mathcal{H}_{\beta\gamma} - N_\beta\mathcal{H})(\mathcal{H} - N^2h^{\alpha\delta}\partial_n h_{\alpha\delta}) + \frac{\epsilon\mathbf{b}}{2N}(N^\alpha\mathcal{H}_{\alpha\beta} - N_\beta\mathcal{H})\partial_n\phi \\
&\quad +\frac{\mathbf{b}}{N}N^\alpha(N_\alpha\partial_n N_\beta - N_\beta\partial_n N_\alpha)\partial_n\phi \\
&\quad +\frac{\epsilon}{2N}(N_\alpha\mathcal{H}_{\beta\gamma} - N_\beta\mathcal{H}_{\alpha\gamma})\partial_n h^{\alpha\gamma} \\
&\qquad\qquad -\frac{1}{2N^3}N^\alpha(N_\alpha\mathcal{H}_{\beta\gamma} - N_\beta\mathcal{H}_{\alpha\gamma})h^{\gamma\delta}\partial_n N_\delta. \qquad (61)
\end{aligned}$$

.

5 Concerning Cauchy Problem and Hamiltonian Formulation

The Cauchy problem in general relativity we are interested in starts from certain data on an initial n-1 dimensional space-like surface to look for their subsequent evolution with the aid of Einstein field equation (for simplicity, in vacuum). Obviously we now have $\epsilon = -1$. Einstein equation includes evolution equation and constraint equation. To solve the field equations we have to find the constraint equations first. For metric (15,17) we easily write down the following n constraint equations

$$(N_\alpha \, N^\alpha + \epsilon \, N^2) R_{nn} - N^\beta \, R_{\beta \, n} - \frac{1}{2} R = 0, \tag{62}$$

$$(N_\alpha \, N^\alpha + \epsilon \, N^2) R_{\beta \, n} - N^\alpha \, R_{\alpha\beta} = 0 \tag{63}$$

because they involve no second time derivatives. Eq.(62) can be simply related to Gauss-Codazzi equations. It can be reduced to

$$\frac{1}{2}(R - 2\epsilon \, R_{ab} n^a n^b) + \frac{\epsilon}{N} N^\beta \, R_{ab} h^a_\beta n^b = 0. \tag{64}$$

However, the relation between other n-1 constraint equations and Gauss-Codazzi equations is not so obvious. Besides, in derivation of Gauss-Codazzi equations we have used only the n-1 dimensional harmonic conditions. To fix the "gauge" it seems necessary to find n "gauge"(coordinate) conditions.
As for the Hamiltonian formulation, presently due to that not only the time derivative $\dot{h}_{\alpha\beta} (\equiv \partial_n h_{\alpha\beta})$, but also $\dot{N}$ or $\dot{\phi}$ and $\dot{N}_\alpha$ appear in Lagrangian

$$\mathcal{L} = \sqrt{-g} R = \frac{\sqrt{h}}{N} R. \tag{65}$$

The problem becomes much more complicated. So we prefer to study it later.

6 Kaluza Klein Reduction

6.1 Neglect the Dependence of Reduced Dimension

In nowadays string theorists think that the four dimensional space-time physics is reduced from a 11-dimensional M-theory through Kaluza-Klein mechanism. Most naturally the reduced dimensions are space-like, so we use $\epsilon = 1$.
When the compactifying radius is very small, the corresponding massive fields can be neglected we then assume that the metric is independent of reduced dimensions, and

$$\tilde{\nabla}_\gamma \, h_{\alpha\beta} = 0, \qquad \tilde{\nabla}_\gamma \, g_{AB} = 0 \tag{66}$$

operator $\tilde{\nabla}_\gamma$ is equivalent to ∇_γ, Gauss-Codazzi equations become quite simple. In fact eq.(49)eq.(50) and eq.(52)are changed into following three equations respectively

$$h^\zeta_C\, h^A_\alpha h^B_\beta h^D_\gamma R_{ADB}{}^C = \mathbf{R}_{\alpha\gamma\beta}{}^\zeta - \frac{1}{2N^2}\mathcal{F}_\beta{}^\zeta \mathcal{F}_{\alpha\gamma} + \frac{1}{4N^2}\mathcal{F}_{\gamma\beta}\mathcal{F}_\alpha{}^\zeta - \frac{1}{4N^2}\mathcal{F}_{\alpha\beta}\mathcal{F}_\gamma{}^\zeta \tag{67}$$

$$h^{AB}h^{CD}R_{ADBC} = R - 2R_{ab}n^a n^b = \mathbf{R} - \frac{3\epsilon}{4N^2}\mathcal{F}_{\alpha\gamma}\mathcal{F}^{\alpha\gamma} \tag{68}$$

$$R = \mathbf{R} - 2\mathbf{b}(\partial^\alpha\partial_\alpha\phi + \mathbf{b}\partial^\alpha\phi\partial_\alpha\phi) - \frac{\epsilon}{4N^2}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}. \tag{69}$$

And eq.(61) is simplified to

$$\tilde{\nabla}_\alpha\, K_\beta{}^\alpha + 2\mathbf{b}K_{\alpha\beta}\partial^\alpha\phi = -R_{BA}h^B_\beta n^A \tag{70}$$

$$= -N(R_{\beta\, n} - N_\beta\, R_{nn}) \quad (K_{\alpha\beta} = \frac{-1}{2N}\mathcal{F}_{\alpha\beta}, \quad N_\beta = \mathcal{A}_\beta, N = e^{-\mathbf{b}\phi}) \tag{71}$$

By use of the "vielbein" method or direct calculation the equivalent form of Gauss equation, the reduction formula, has already presented in literature (see for example [8][9][10] and references therein).

6.2 Combine With the Fiber Bundle

Gauss-Codazzi equations can be set up in distinct regions (exactly, neighborhoods) and there is a transition function matching them up. As an example let us examine the 11 dimensional Kaluza-Klein monopole

$$ds^2_{11} = h_{\alpha\beta}dx^\alpha\, dx^\beta + \frac{1}{N^2}(dx^{10} + \mathcal{A}^\pm)^2 \tag{72}$$

in which

$$N = e^{-\frac{4}{3}\phi}$$

is a functional of dilaton field ϕ. Set $x_\alpha = (x_\mu, y_i)$, $\mu = 0, \cdots, 7$; $\quad i = 1, 2, 3$

$$\mathcal{A}^\pm = \frac{Q_m}{2r(y_3 + r)}(-y_2 dy_1 + y_1 dy_2) = \frac{1}{2}Q_m(\pm 1 - \cos\theta)d\phi \tag{73}$$

($r = \sqrt{y_i y_i}$) is the monopole in "south" and "north" regions (semisphere) respectively [11]. Therefore we get in these respective regions, Gauss and Codazzi equations with $N_\alpha = \mathcal{A}^+_\alpha$ and $\mathcal{A}^-_\alpha$. The transition function is the well-known gauge transformation from $\mathcal{A}^\mp$ to $\mathcal{A}^\pm$. Of course, when we neglect the dependence of the metric on reduced dimensions, the difference in Gauss equation is

trivial (because that Gauss equation depends only on a functional of $\mathcal{A}, \mathcal{F}$, but not on $\mathcal{A}$ itself).
By a conformal transformation the 10 dimensional part of eq.(72) is a D6 brane metric (in string frame)

$$ds_{10}^2 = N^{-1} h_{\alpha\beta} dx^\alpha\, dx^\beta = N^{-1} \eta_{\mu\nu} dx^\mu\, dx^\nu + N dy^i dy^i, \tag{74}$$

and

$$\mathcal{F}_2 = e^{-\frac{3}{2}\phi} * (dx^7 \wedge dN^{-2}) \tag{75}$$

$$= \frac{1}{2r^3} \epsilon_{ijk} y^i dy^j\, dy^k = d\mathcal{A}^{\pm}. \tag{76}$$

6.3 Reduced dimensions larger than 1

Because of the complexity we will continue the assumption that the metric is independent of reduced dimensions Thus we can easily prove that equations

$$\tilde{\nabla}_\gamma\, h_{\alpha\beta} = 0, \qquad \tilde{\nabla}_\gamma\, g_{AB} = 0$$

are still hold in higher reduced dimensions. So we can use g_{AB} and $h_{\alpha\beta}$ to raise and lower indices of whole space and subspace respectively. And we may omit tilde symbol on the covariant derivative operator. Let us start from a metric

$$ds^2 = g_{AB} dx^A dx^B = h_{\alpha\beta} dx^\alpha\, dx^\beta + N_{ij}(dy^i + N^i_\alpha dx^\alpha)(dy^j + N_\beta\, dx^\beta) \tag{77}$$

$$N_{ij} = N_{ji} \tag{78}$$

in which we have used $i, j, \cdots$ to denote the indices of reduced dimensions, and they are Euclidean. Then, the metric has the form

$$g_{AB} = h_{AB} + N_{ij} n^i_A n^j_B. \qquad A = (\alpha, j) \tag{79}$$

where

$$n^i_A = (N^i_\alpha, \delta^i_j), \qquad n^{iA} = (0, N^{-1\, ij}) \tag{80}$$

and

$$n^i_A n^A_j = N^{-1}{}_j{}^i, \qquad h_{AB} n^B_i = 0 = h^{AB} n^i_B, \tag{81}$$

explicitly we have for example

$$h^{AB} = \begin{pmatrix} h^{\alpha\beta} & -N^{j\alpha} \\ -N^{i\beta} & N^i_\alpha N^{j\alpha} \end{pmatrix}, \qquad N^j_i n^A_j n^i_B = \begin{pmatrix} 0 & 0 \\ N^i_\beta & \delta^i_j \end{pmatrix}. \tag{82}$$

Along the same line as in the derivation of eq.(27) the Gauss formula becomes

$$\nabla_\alpha\, h^A_\beta = K_{\alpha\beta}{}^i n^A_i, \tag{83}$$

where

$$K_{\alpha\beta}{}^i \equiv (\partial_\alpha\, h^D_\beta + \Gamma^D_{BC} h^B_\alpha h^C_\beta) n^j_D N^i_j \equiv \tilde{K}_{\alpha\beta}{}^j N^i_j. \tag{84}$$

Because that

$$h^A_\beta \nabla_\alpha n^i_A = h^A_\beta (\partial_\alpha n^i_A - \Gamma^C_{BA} h^B_\alpha n^i_C), \tag{85}$$

$$= -(\partial_\alpha h^A_\beta + \Gamma^A_{BC} h^B_\alpha h^C_\beta) n^i_A = -K_{\alpha\beta}{}^j n^A_j n^i_A = -\tilde{K}_{\alpha\beta}{}^i, \tag{86}$$

and

$$- h^\beta_D K_{\alpha\beta}{}^i = (\delta^A_D - N^k_j n^A_k n^j_D) \nabla_\alpha n^i_A = \nabla_\alpha n^i_D - n^j_D L_{\alpha\, j}{}^i, \tag{87}$$

here we have defined

$$L_{\alpha\, j}{}^i \equiv N^k_j n^A_k \nabla_\alpha n^i_A \equiv N^k_j \tilde{L}^i_{\alpha\, k}. \tag{88}$$

Therefore the Weingarten formula can be written as

$$\nabla_\alpha n^i_A = n^j_A L_{\alpha\, j}{}^i - h^\beta_A K_{\alpha\beta}{}^j N^{-1}{}_j{}^i = -h^\beta_A \tilde{K}_{\alpha\beta}{}^i + N^j_k n^k_A \tilde{L}_{\alpha\, j}{}^i. \tag{89}$$

Since

$$(\nabla_\alpha n^i_A) n^A_j + n^i_A \nabla_\alpha n^A_j =$$
$$-h^\beta_A \tilde{K}_{\alpha\beta}{}^i n^A_j + N^l_k n^k_A \tilde{L}_{\alpha\, l}{}^i n^A_j - n^i_A h^{A\gamma} \tilde{K}_{\alpha\gamma\, j} + n^i_A N^l_k \tilde{L}_\alpha{}^k{}_j n^A_l =$$

$$\tilde{L}_{\alpha\, j}{}^i + \tilde{L}_\alpha{}^i{}_j. \tag{90}$$

So the tensor L satisfies the relation

$$\tilde{L}_{\alpha\, ji} + \tilde{L}_{\alpha\, ij} = \partial_\alpha N^{-1}{}_{ij}. \tag{91}$$

In the following we will derive the Gauss equation. Since

$$\nabla_\alpha \nabla_\beta h^A_\gamma - \nabla_\beta \nabla_\alpha h^A_\gamma = h^C_\alpha h^D_\beta h^B_\gamma R^A{}_{BCD} - h^A_\delta \mathbf{R}^\delta{}_{\gamma\alpha\beta} + \Gamma^A_{BC} (\partial_\alpha h^B_\beta - \partial_\beta h^B_\alpha) h^C_\gamma, \tag{92}$$

and

$$h^\delta_A \nabla_\alpha \nabla_\beta h^A_\gamma = -(\nabla_\alpha h^\delta_A)(\nabla_\beta h^A_\gamma) \tag{93}$$

$$= N^{-1i}{}_j K_\alpha{}^\delta{}_i K_{\beta\gamma}{}^j = -N^i_j \tilde{K}_\alpha{}^\delta{}_i \tilde{K}_{\beta\alpha}{}^j, \tag{94}$$

hence

$$h^\delta_A h^C_\alpha h^D_\beta h^B_\gamma R^A{}_{BCD} = \mathbf{R}^\delta{}_{\gamma\alpha\beta} - N^{-1i}{}_j (K_\alpha{}^\delta{}_i K_{\beta\gamma}{}^j - K_\beta{}^\delta{}_i K_{\alpha\gamma}{}^j) + (\Gamma^\delta_{i\gamma} - N^j_\gamma \Gamma^\delta_{ij}) \mathcal{F}_{\alpha\beta}{}^i \tag{95}$$

in which

$$\mathcal{F}_{\alpha\beta}{}^i = \partial_\alpha N^i_\beta - \partial_\beta N^i_\alpha, \tag{96}$$

and

$$\Gamma^\delta_{i\gamma} - N^j_\gamma \Gamma^\delta_{ij} = \frac{1}{2}[g^{\delta\alpha}(\partial_\gamma g_{i\alpha} - \partial_\alpha g_{i\gamma} + N^j_\gamma \partial_\alpha g_{ij}) + g^{\delta\, j} \partial_\gamma g_{ij}] = \frac{1}{2} N_{ij} h^{\delta\alpha} \mathcal{F}_{\gamma\alpha}{}^j. \tag{97}$$

At the end we obtain Gauss equation

$$h_\alpha^A h_\beta^B h_\gamma^C h_\delta^D R_{ABCD} = \mathbf{R}_{\alpha\beta\gamma\delta} + N^{-1}{}^i_j (K_{\alpha\delta\, i} K_{\beta\gamma}{}^j - K_{\beta\delta\, i} K_{\alpha\gamma}{}^j) - \frac{1}{2} N_{ij} \mathcal{F}_{\alpha\beta}{}^i \mathcal{F}_{\gamma\delta}{}^j. \tag{98}$$

To get Codazzi equation and Ricci equation we note that the following equation is available

$$\nabla_\beta \nabla_\alpha\, n_i^A - \nabla_\alpha \nabla_\beta\, n_i^A = R^A{}_{BDC} h_\alpha^C h_\beta^D n_i^B + \Gamma^A_{BC} (\partial_\beta\, h_\alpha^B - \partial_\alpha\, h_\beta^B) n_i^C. \tag{99}$$

By using of Weingarten formula we know

$$\nabla_\beta \nabla_\alpha\, n_i^A = -(\nabla_\beta \tilde{K}_\alpha{}^\gamma{}_i) h_\gamma^A - \tilde{K}_\alpha{}^\gamma{}_i K_{\beta\gamma}{}^j n_j^A + (\nabla_\beta\, L_\alpha{}^j{}_i) n_j^A + L_\alpha{}^j{}_i (-\tilde{K}_\beta{}^\gamma{}_j h_\gamma^A + L_\beta{}^k{}_j n_k^A), \tag{100}$$

thus

$$\begin{aligned} (\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) n_i^A = & \\ & -(\nabla_\beta\, K_\alpha{}^\gamma{}_i - \nabla_\alpha \tilde{K}_\beta{}^\gamma{}_i) h_\gamma^A - (\tilde{K}_\alpha{}^\gamma{}_i K_{\beta\gamma}{}^j - \tilde{K}_\beta{}^\gamma{}_i K_{\alpha\gamma}{}^j) n_j^A \\ & +(\nabla_\beta\, L_\alpha{}^j{}_i - \nabla_\alpha\, L_\beta{}^j{}_i) n_j^A - L_\alpha{}^j{}_i \tilde{K}_\beta{}^\gamma{}_j - L_\beta{}^j{}_i \tilde{K}_\alpha{}^\gamma{}_j) h_\gamma^A + \end{aligned}$$

$$(L_\alpha{}^j{}_i L_\beta{}^k{}_j - L_\beta{}^j{}_i L_\alpha{}^k{}_j) n_k^A \tag{101}$$

$$= R^A{}_{BCD} h_\beta^C h_\alpha^D n_i^B + \Gamma^A_{BC} (\partial_\beta\, h_\alpha^C - \partial_\alpha\, h_\beta^C) n_i^B. \tag{102}$$

Note that

$$\Gamma^\gamma_{jk} = -\frac{1}{2} \partial^\gamma\, g_{jk}, \qquad \Gamma^l_{kj} = \frac{1}{2} N^l_\gamma \partial^\gamma\, g_{jk}, \tag{103}$$

at last we obtain Codazzi equation

$$\nabla_\alpha \tilde{K}_{\beta\gamma\, i} - \nabla_\beta \tilde{K}_{\alpha\gamma\, i} - L_\alpha{}^j{}_i \tilde{K}_{\beta\gamma\, j} + L_\beta{}^j{}_i \tilde{K}_{\alpha\gamma\, j} = -R_{ABCD} h_\alpha^C h_\beta^D n_i^B - \frac{1}{2} (\partial_\gamma\, N_{jk}) N^{-1}{}^k_i \mathcal{F}^j_{\alpha\beta}, \tag{104}$$

and Ricci equation

$$\begin{aligned} & (\nabla_\beta\, L_\alpha{}^j{}_i - \nabla_\alpha\, L_\beta{}^j{}_i) N^{-1}{}^l_j + (L_\alpha{}^j{}_i L_\beta{}^k{}_j - L_\beta{}^j{}_i L_\alpha{}^k{}_j) N^{-1}{}^l_k - \\ & (\tilde{K}_\alpha{}^\gamma{}_i K_{\beta\gamma}{}^j - \tilde{K}_\beta{}^\gamma{}_i K_{\alpha\gamma}{}^j) N^{-1}{}^l_j = \end{aligned}$$

$$R^A{}_{BCD} h_\beta^C h_\alpha^D n_i^B n_A^l - \frac{1}{2} (\delta_\delta^\gamma - N_\delta^l)(\partial^\delta\, N_{jk}) N^{-1}{}^k_i \mathcal{F}_{\alpha\beta}{}^j. \tag{105}$$

7 Discussion

In this paper we have derived variations of Gauss-Codazzi-Ricci equations, but there are several questions we need to research further.Firstly, we want to know what geometric meaning the tensor $K_{\alpha\beta}$ has? Ordinary, the corresponding tensor presented in Gauss-Codazzi-Ricci equations is the extrinsic curvature, which is connected with the second fundamental form. In three dimensional Euclidean space half of second fundamental form is the principle part of the departure from tangent plane to a point which is in the neighborhood of the tangent point on the surface .
Next, as mentioned in sec.5 we have to perform the Hamiltonian formulation of general relativity in new metric construction.
Thirdly, if metric depends on reduced dimensions in some special way, such as the spherical reductions [12] it must be very interesting to know what new feature will arise.

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[12] see for example M.J.Duff, C.N.Pope, Kaluza-Klein supergravity and the seven sphere, in Supersymmetry and Supergravity (S.Ferrara, J.G.Taylor, P.van Nieuwenhuizen,eds.), World Scientific, Singapore;1983.

Christoffel Symbols for One Reduced Dimension

$$\Gamma^{\gamma}_{\alpha\beta} = \mathbf{\Gamma}^{\gamma}_{\alpha\beta} + \frac{\epsilon\, h^{\gamma\delta}}{2N^2}(N_\alpha \mathcal{F}_{\beta\delta} + N_\beta \mathcal{F}_{\alpha\delta} - 2\mathbf{b} N_\alpha\, N_\beta \partial_\delta \phi) + \frac{N^\gamma}{2N^2}(\mathcal{H}_{\alpha\beta} + 2\epsilon \mathbf{b} N_\alpha\, N_\beta \partial_n \phi) \tag{106}$$

$$\Gamma^{n}_{\alpha\beta} =$$
$$-N_\gamma \mathbf{\Gamma}^{\gamma}_{\alpha\beta} - \frac{\epsilon\, N^\gamma}{2N^2}(N_\alpha \mathcal{F}_{\beta\gamma} + N_\beta \mathcal{F}_{\alpha\gamma} - 2\mathbf{b} N_\alpha\, N_\beta \partial_\gamma \phi) + 1/2N^2[\partial_\alpha (N_\beta / N^2)$$
$$+\partial_\beta (N_\alpha / N^2)] - 1/2N^2 (N_\gamma\, N^\gamma + \epsilon\, N^2)(\mathcal{H}_{\alpha\beta} + 2\epsilon \mathbf{b} N_\alpha\, N_\beta \partial_n \phi) \tag{107}$$

$$\Gamma^{\beta}_{n\alpha} = \frac{h^{\beta\gamma}}{2N^2}[\epsilon(\mathcal{F}_{\alpha\gamma} - 2\mathbf{b} N_\alpha \partial_\gamma \phi) + \mathcal{H}_{\gamma\alpha} + 2\epsilon \mathbf{b} N_\gamma\, N_\alpha \partial_n \phi] \tag{108}$$

$$\Gamma^{n}_{n\alpha} = -\frac{\epsilon\, N^\gamma}{2N^2}(\mathcal{F}_{\alpha\gamma} - 2\mathbf{B} N_\alpha \partial_\gamma \phi) + \mathbf{b}\partial_\alpha \phi - \frac{N^\gamma}{2N^2}(\mathcal{H}_{\gamma\alpha} + 2\epsilon \mathbf{b} N_\gamma\, N_\alpha \partial_n \phi) \tag{109}$$

$$\Gamma^{\gamma}_{nn} = \frac{\epsilon}{N^2}[-\mathbf{b}(\partial^\gamma \phi - N^\gamma \partial_n \phi) + h^{\gamma\alpha}\partial_n N_\alpha] \tag{110}$$

$$\Gamma^{n}_{nn} = \frac{\epsilon}{N^2}[\mathbf{b}(N_\gamma \partial^\gamma \phi - (N_\gamma\, N^\gamma - \epsilon\, N^2)\partial_n \phi) - N^\gamma \partial_n N_\gamma] \tag{111}$$

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